SMOOTH PATHS OF CONDITIONAL EXPECTATIONS*

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Abstract

Let $\mathcal{A}$ be a von Neumann algebra with a finite trace τ, represented in $\mathcal{H} = L^2(\mathcal{A}, \tau)$, and let $\mathcal{B}_t \subset \mathcal{A}$ be sub-algebras, for t in an interval I $(0 \in I)$. Let $E_t : \mathcal{A} \to \mathcal{B}_t$ be the unique τ-preserving conditional expectation. We say that the path $t \mapsto E_t$ is smooth if for every $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, the map

$$I \ni t \mapsto E_t(a)\xi \in \mathcal{H}$$

is continuously differentiable. This condition implies the existence of the derivative operator

$$dE_t(a) : \mathcal{H} \to \mathcal{H},\ dE_t(a)\xi = \frac{d}{dt}E_t(a)\xi.$$

If this operator verifies the additional boundedness condition,

$$\int_J \|dE_t(a)\|_2^2 dt \leq C_J \|a\|_2^2,$$

for any closed bounded sub-interval $J \subset I$, and $C_J > 0$ a constant depending only on J, then the algebras $\mathcal{B}_t$ are $*$-isomorphic. More precisely, there exists a curve $G_t : \mathcal{A} \to \mathcal{A}$, $t \in I$ of unital, $*$-preserving linear isomorphisms which intertwine the expectations,

$$G_t \circ E_0 = E_t \circ G_t.$$

The curve G_t is weakly continuously differentiable. Moreover, the intertwining property in particular implies that G_t maps $\mathcal{B}_0$ onto $\mathcal{B}_t$. We show that this restriction is a multiplicative isomorphism. [1]

1 Introduction

Let $\mathcal{A}$ be a von Neumann algebra with a finite faithful and normal trace τ, and suppose $\mathcal{A}$ acting on its standard Hilbert space $\mathcal{H} = L^2(\mathcal{A}, \tau)$. We shall assume that for each $t \in I$ $(0 \in I)$, there is a von Neumann sub-algebra $\mathcal{B}_t \subset \mathcal{A}$, and we shall denote by $E_t : \mathcal{A} \to \mathcal{B}_t$ the unique τ-invariant conditional expectation. We regard $t \mapsto E_t$ as a curve, and require smoothness in the following sense: for each $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, $I \ni t \mapsto E_t(a)\xi \in \mathcal{H}$ is continuously differentiable. This paper is a sequel to [1], where a similar matter is treated with more strict hypothesis. In [1] we considered a stronger smoothness condition, namely, that for each $a \in \mathcal{A}$, the map $I \ni t \mapsto E_t(a) \in \mathcal{A}$ is continuously differentiable (in norm).
The current regularity assumption on E_t implies the existence of the bounded derivative operator, for each $t \in I$ and $a \in \mathcal{A}$

$$dE_t(a) : \mathcal{H} \to \mathcal{H},\quad dE_t(a)\xi = \frac{d}{dt}E_t(a)\xi.$$

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Therefore a curve of possibly unbounded symmetric operators dE_t is defined in $\mathcal{H}$, with common domain $\mathcal{A} \subset \mathcal{H}$. We shall make the following assumption on dE:

$$\int_J \|dE_t(a)\|_2^2 dt \leq C_J \|a\|_2^2 \tag{1}$$

for all $a \in \mathcal{A}$, and every closed bounded interval $J \subset I$ (the constant depends only on J). With these assumptions, we prove that there exists a curve $I \ni t \mapsto G_t$ of linear isomorphisms $G_t : \mathcal{A} \mapsto \mathcal{A}$ with the following properties:

1. For each $a \in \mathcal{A}$, the curve $I \ni t \to G_t(a) \in \mathcal{A} \subset \mathcal{H}$ is weakly continuously differentiable, with $G_0 = Id$.
2. The maps G_t are unital and $*$-preserving.
3. For each $t \in J_0$,
$$G_t E_0 G_t^{-1} = E_t.$$
4. The last formula implies that G_t maps $\mathcal{B}_0$ onto $\mathcal{B}_t$. The restriction
$$G_t|_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathcal{B}_t$$
is a $*$-isomorphism.
5. The linear isomorphisms $G_t : \mathcal{A} \to \mathcal{A}$ are $\| \ \|_2$-isometric, therefore they extend to unitary operators U_t acting in $\mathcal{H}$, which preserve $\mathcal{A}$ ($U_t(\mathcal{A}) = \mathcal{A}$).

A similar result was obtained in [1] with the already noted stronger assumption. In both contexts, the maps G_t appear as propagators of the linear differential equation

$$\begin{cases} \dot{\alpha}(t) = dE_t(E_t(\alpha(t))) - E_t(dE_t(\alpha(t))) \\ \alpha(s) = a, \end{cases} \tag{2}$$

for $a \in \mathcal{A}$. In the present context, our hypothesis does not guarantee that the linear operators $[dE, E]$ of this equation are bounded, nor that they vary continuously. Therefore our first task is to show that with the current assumptions (particularly 1), this equation has existence and uniqueness of *weak* solutions. This is done in section 3. In section 2 we state the basic properties of the operator dE. In section 4 we prove the existence and properties of the maps G_t. In section 5 we consider the example when the expectations E_t are given by a curve of systems of projections $p_1(t), p_2(t), ...$ in $\mathcal{A}$ (i.e. curves of pairwise orthogonal projections which sum up to 1), and examine when our hypothesis are verified.

2 Curves of expectations

As we stated above, we shall consider $\mathcal{A}$ represented in the standard space $\mathcal{H} = L^2(\mathcal{A}, \tau)$, and also regard elements of a as elements in $\mathcal{H}$. We shall denote by $\| \ \|_\infty$ the norm of $\mathcal{A}$, and by $\| \ \|_2$ the norm of $\mathcal{H}$.

Lemma 2.1. *For each $a \in \mathcal{A}$ and $t \in I$, the linear operator $dE_t(a)$ defined in the previous section is bounded, its adjoint is $dE_t(a^*)$.*

Proof. Note that both $dE_t(a)$ and $dE_t(a^*)$ are defined in the whole space $\mathcal{H}$ by hypothesis. If $x, y \in \mathcal{A}$, regarded as a dense subspace of $\mathcal{H}$,

$$< dE_t(a)x, y >= \frac{d}{dt} < E_t(a)x, y >= \frac{d}{dt}\tau(y^* E_t(a)x) = \frac{d}{dt}\tau((E_t(a^*)y)^* x)$$

$$= \frac{d}{dt} < x, E_t(a^*)y >=< x, dE_t(a^*)y > .$$

By the closed graph theorem, it follows that $dE_t(a)$ is bounded, and that $dE_t(a^*)$ is its adjoint. $\square$

Next let us show that the derivative of E_t defines also a map on $\mathcal{A}$.

Lemma 2.2. *Let $a \in \mathcal{A}$, then for each $t \in I$, $dE_t(a) \in \mathcal{A}$.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ belong to the commutant of $\mathcal{A}$. If $\xi, \eta \in \mathcal{H}$,

$$< dE_t(a)T\xi, \eta >= \frac{d}{dt} < E_t(a)T\xi, \eta >= \frac{d}{dt} < TE_t(a)\xi, \eta >= \frac{d}{dt} < E_t(a)\xi, T^*\eta >$$

$$=< dE_t(a)\xi, T^*\eta >=< TdE_t(a)\xi, \eta >,$$

i.e. $dE_t(a) \in \mathcal{A}$. □

The correspondence $dE_t : \mathcal{A} \to \mathcal{A}$ is apparently linear, and $*$-preserving. Let us verify that it is bounded as an operator acting in $(\mathcal{A}, \| \ \|_\infty)$.

Proposition 2.3. *For each $t \in I$, the map $dE_t : (\mathcal{A}, \| \ \|_\infty) \to (\mathcal{A}, \| \ \|_\infty)$, $a \mapsto dE_t(a)$, is linear, $*$-preserving and bounded. Moreover, for any closed and bounded sub-interval $J \subset I$, the norms of the operators $dE_t : (\mathcal{A}, \| \ \|_\infty) \to (\mathcal{A}, \| \ \|_\infty)$, denoted $\|dE_t\|_{\infty,\infty}$, are uniformly bounded for $t \in J$.*

Proof. Let us prove that the graph of dE_t is closed. Let $a_n, a, b \in \mathcal{A}$ such that $\|a_n - a\|_\infty \to 0$ and $\|dE_t(a_n) - b\|_\infty \to 0$. First note that if $x, y \in \mathcal{A}$, then

$$\tau(dE_t(x)y) = \tau(xdE_t(y)).$$

Indeed, by the invariance of E_t and τ,

$$\tau(E_t(x)y) = \tau(E_t(E_t(x)y)) = \tau(E_t(x)E_t(y)) = \tau(E_t(xE_t(y))) = \tau(xE_t(y)).$$

Then

$$\tau(dE_t(x)y) =< dE_t(x), y^* >= \frac{d}{dt} < E_t(x), y^* >= \frac{d}{dt}\tau(E_t(x)y) = \frac{d}{dt}\tau(xE_t(y)),$$

which by the same argument equals $\tau(xdE_t(y))$. Therefore, for any $x \in \mathcal{A}$,

$$\tau(bx) = \lim_{n\to\infty} \tau(dE_t(a_n)x) = \lim_{n\to\infty} \tau(a_n dE_t(x)) = \tau(a \ dE_t(x)) = \tau(dE_t(a)x).$$

It follows that $dE_t(a) = b$, and therefore dE_t is bounded.
Consider now a closed bounded sub-interval $J \subset I$. Fix $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$. Since by hypothesis the map $t \mapsto E_t(a)\xi$ is continuously differentiable, it follows that there exists a constant $C_{J,a,\xi}$ such that

$$\|dE_t(a)\xi\|_2 \leq C_{J,a,\xi} \quad \text{for all } t \in J.$$

By the uniform boundedness principle in the Banach space $(\mathcal{H}, \| \ \|_2)$, it follows that there exists a constant $C_{J,a}$ such that

$$\|dE_t(a)\|_\infty \leq C_{J,a} \quad \text{for all } t \in J.$$

Again by the uniform boundedness principle, this time in the Banach space $(\mathcal{A}, \| \ \|_\infty)$, it follows that there exists a constant C_J such that

$$\|dE_t\|_{\infty,\infty} \leq C_J \quad \text{for all } t \in J.$$

□

We emphasize that dE_t may be an unbounded operator in $\mathcal{H}$, with domain $\mathcal{A}$.

Remark 2.4. The assumption that $I \ni t \mapsto E_t(a)\xi \in \mathcal{H}$ is continuously differentiable implies that $t \mapsto E_t(a) \in \mathcal{H}$ is continuously differentiable. Indeed, it suffices to take $\xi = 1 \in \mathcal{A}$.

We shall need the following elementary fact.

Lemma 2.5. *For $h \in [-\delta, \delta]$, let $b_h, b \in \mathcal{A}$ such that $\|b_h - b\|_2 \to 0$ as $h \to 0$. Then*

$$\|E_{t+h}(b_h) - E_t(b)\|_2 \to 0 \quad \text{as } h \to 0.$$

Proof. Note that

$$\|E_{t+h}(b_h) - E_t(b)\|_2 \leq \|E_{t+h}(b_h) - E_{t+h}(b)\|_2 + \|E_{t+h}(b) - E_t(b)\|_2.$$

The second term clearly tends to 0. Since the expectations E_t are τ-invariant, they are contractive for the $\| \ \|_2$-norm. Therefore the first term is bounded by $\|b_h - b\|_2$. □

We shall use the following formula thoroughly.

Proposition 2.6. *For any $a \in \mathcal{A}$ and any $t \in I$,*

$$dE_t(E_t(a)) + E_t(dE_t(a)) = dE_t(a).$$

Proof. Note that

$$\frac{1}{h}\{E_{t+h}(a) - E_t(a)\} = \frac{1}{h}\{E_{t+h}(E_{t+h}(a)) - E_t(E_t(a))\}$$

$$= E_{t+h}(\frac{1}{h}\{E_{t+h}(a) - E_t(a)\}) + \frac{1}{h}\{E_{t+h}(E_t(a)) - E_t(E_t(a))\}.$$

The second term tends to $dE_t(E_t(a))$ in the 2-norm. The first term tends to $E_t(dE_t(a))$ in the 2-norm by the above Lemma, which proves the formula. □

3 The transport equation

Under the current assumptions we shall examine existence and uniqueness of solutions of the linear differential equation below, which we shall call the transport equation (2)

$$\begin{cases} \dot{\alpha}(t) = dE_t(E_t(\alpha(t))) - E_t(dE_t(\alpha(t))) \\ \alpha(s) = a, \end{cases}$$

where $a \in \mathcal{A}$. We shall be looking for solutions $\alpha(t)$ with values in $\mathcal{A}$, which are differentiable as $\mathcal{H}$-valued maps in the weak sense. That is, $t \mapsto < \alpha(t), \xi >$ is differentiable, and its derivative verifies

$$\frac{d}{dt} < \alpha(t), \xi > = < dE_t(E_t(\alpha(t))) - E_t(dE_t(\alpha(t))), \xi >,$$

for all $\xi \in \mathcal{H}$.
Note that the classical results on linear differential equations in Banach spaces (for instance, [2, 3]) do not apply. The linear operators $[dE_t, E_t]$ need not be continuous in the parameter t as operators in the Banach space $\mathcal{A}$, nor they need to be bounded as operators in $\mathcal{H}$ (with common domain $\mathcal{A}$), or even closed operators. This seems to be a mixed terrain, where both considerations with the non equivalent norms $\| \ \|_\infty$ and $\| \ \|_2$ play a role. We shall show existence and uniqueness of solutions mimicking carefully Picard's method of succesive approximations, under the assumption of the following Hypothesis (1):

$$\int_J \|dE_t(a)\|_2^2 dt \leq C_J \|a\|_2^2$$

for all $a \in \mathcal{A}$, and every closed bounded interval $J \subset I$ (the constant depends only on J). Note that this hypothesis trivially holds if dE is bounded in the 2-norm $\| \ \|_2$. Indeed, this holds by the uniform boundedness principle.
We shall mainly be involved with the properties of the operator $H_t = [dE_t, E_t]$. Note that $H_t(\mathcal{A}) \subset \mathcal{A}$. Also it is clear that H_t is anti-symmetric in $\mathcal{A}$: if $x, y \in \mathcal{A}$ then

$$< H_t(x), y >=< dE_t(E_t(x)), y > - < E_t(dE_t(x)), y >$$

$$=< x, E_t(dE_t(y) > - < x, dE_t(E_t(y)) >= - < x, H_t(y) > .$$

Also it is apparent that for each fixed $x \in \mathcal{A}$, $t \mapsto H_t(x) \in \mathcal{H}$ is continuous.
The following result will be needed. It is not supposed in the next Lemma that E_t verifies Hypothesis (1).

Lemma 3.1. *Let $f : I \to \mathcal{A}$ be uniformly $\| \ \|_\infty$-bounded on closed bounded sub-intervals of I, and weakly continuous when regarded as an $\mathcal{H}$-valued map, i.e.*

1. *For every closed bounded $J \subset I$ there exists a constant C_J such that $\|f(t)\|_\infty \le C_J$ for all $t \in J$.*
2. *For every $\xi \in \mathcal{H}$, the map $t \mapsto< f(t), \xi >$ is continuous.*

Then the map $t \mapsto H_t(f(t))$ takes values in $\mathcal{A}$, is weakly continuous as an $\mathcal{H}$-valued map, and is uniformly $\| \ \|_\infty$-bounded on closed bounded intervals as an $\mathcal{A}$-valued map.

Proof. First pick $x \in \mathcal{A}$. Then $g_x(t) =< H_t(f(t)), x >= - < f(t), H_t(x) >$. Thus

$$g_x(t+h) - g_x(t) = - < f(t+h), H_{t+h}(x) > + < f(t), H_t(x) >$$

$$=< f(t+h), H_t(x) - H_{t+h}(x) > + < f(t+h) - f(t), H_t(x) > .$$

The second term tends to 0 as $h \to 0$. By the Cauchy-Scwarz inequality, the first term is bounded by

$$\|f(t+h)\|_2 \|H_{t+h}(x) - H_t(x)\|_2.$$

This expression also tends to 0, as $h \to 0$, because f is $\| \ \|_\infty$ bounded (and therefore also $\| \ \|_2$ bounded). Let $\xi \in \mathcal{H}$ and pick $x \in \mathcal{A}$ such that $\|\xi - x\|_2 < \epsilon$. Then if $g_\xi(t) =< H_t(f(t)), \xi >$,

$$g_\xi(t+h) - g_\xi(t) =< H_{t+h}(f(t)), \xi - x > + g_x(t+h) - g_x(t) + < H_t(f(t)), x - \xi > .$$

If $h \to 0$, the middle term tends to 0. Again, by the Cauchy-Scwarz inequality, the first term is bounded by

$$\|H_{t+h}(f(t+h))\|_2 \|\xi - x\|_2 \le \|H_{t+h}(f(t+h))\|_\infty \|\xi - x\|_2 \le \epsilon \|H_{t+h}\|_{\infty,\infty} \|f(t+h)\|_\infty.$$

For small h (e.g. $|h| \le \delta$ such that $J = [t-\delta, t+\delta] \subset I$), both factors above are uniformly bounded. For instance $\|H_t\|_{\infty,\infty} \le 2\|dE_t\|_{\infty,\infty}$, and then use Proposition 2.3. The third term is dealt similarly. This proves the weak continuity of $t \mapsto H_t(f(t)) \subset \mathcal{H}$.
Local boundedness in $\| \ \|_\infty$ is straightforward: $\|H_t(f(t))\|_\infty \le 2\|dE_t\|_{\infty,\infty} \|f(t)\|_\infty$. □

Fix $a \in \mathcal{A}$ and s in the interior of I. For each $t \in I$, consider the following sequence of functions $S_n^{a,s}(t) = S_n(t)$:

Definition 3.2.

$$S_0(t) = a \ , \quad S_1(t) = a + \textbf{weak} \int_s^t H_u(a) du, \quad \textit{and} \quad S_{n+1}(t) = a + \textbf{weak} \int_s^t H_u(S_n(u)) du,$$

where **weak** $\int$ *stands for the weak integral, i.e. for each $\xi \in \mathcal{H}$,* **weak** $\int_J f(u)du$ *is given by*

$$< \textbf{weak} \int_J f(u) du, \xi >= \int_J < f(u), \xi > du.$$

First we must show that $S_n(t)$ is well defined.

Proposition 3.3. *For any fixed $a \in \mathcal{A}$ and s in the interior of I, the maps $S_n(t)$, $t \in I$ are well defined. They take values in $\mathcal{A}$. Regarded as $\mathcal{A}$-valued functions, they are uniformly bounded on closed bounded sub-intervals of I. Regarded as $\mathcal{H}$-valued functions, they are weakly continuous.*

Proof. This is proved by induction. Clearly S_0 takes values in $\mathcal{A}$, is $\| \|_\infty$-bounded uniformly bounded on closed bounded intervals, and is $\mathcal{H}$-weakly continuous. Suppose that S_n verifies these conditions. By the above lemma, the map $t \mapsto H_t(S_n(t))$ is $\mathcal{H}$-weakly continuous and $\| \|_\infty$-bounded. Therefore, it only remains to be verified that it takes values in $\mathcal{A}$. The weak integral $\int_s^t H_u(S_n(u))du$ is the weak limit of its Riemann sums $\sum_j H_{u_j}(S_n(u_j))(u_j - u_{j-1})$, which are linear combinations of elements of $\mathcal{A}$, and thus lie in $\mathcal{A}$. Moreover

$$\| \sum_j H_{u_j}(S_n(u_j))(u_j - u_{j-1}) \|_\infty \leq \sum_j \|H_{u_j}(S_n(u_j))\|_\infty (u_j - u_{j-1}).$$

Each term $\|H_{u_j}(S_n(u_j))\|_\infty$ is uniformly bounded in the interval $[s,t]$. Therefore the Riemann sums are uniformly $\| \|_\infty$-bounded. Therefore the weak limit of these sums lies in $\mathcal{A}$. $\square$

For the next result we need Hypothesis (1)

Proposition 3.4. *Fix $s_0 \leq t_0$ in I and $a \in \mathcal{A}$, and consider $S_n(t) = S_n^{s_0,a}(t)$. Assume that Hypothesis (1) holds: $\int_{s_0}^{t_0} \|dE_s(b)\|_2^2 ds \leq C\|b\|_2^2$ (where $C = C_{[s_0,t_0]}$). Then for all $t \in [s_0, t_0]$,*

$$\|S_{n+1}(t) - S_n(t)\|_2 \leq C^{1/2}\sqrt{t - s_0} \sup_{u \in [s_0,t]} \|S_n(u) - S_{n-1}(u)\|_2.$$

Proof. Pick $b \in \mathcal{A}$. Then

$$| < S_{n+1}(t) - S_n(t), b > | = | \int_{s_0}^t < H_u(S_n(u)) - H_u(S_{n-1}(u)), b > du|$$

$$= | \int_{s_0}^t < S_n(u) - S_{n-1}(u), H_u(b) > du| \leq \int_{s_0}^t | < S_n(u) - S_{n-1}(u), H_u(b) > |du$$

$$\leq \sup_{u \in [s_0,t]} \|S_n(u) - S_{n-1}(u)\|_2 \int_{s_0}^t \|H_u(b)\|_2 du.$$

By Hölder's inequality

$$\int_{s_0}^t \|H_u(b)\|_2 du \leq \{ \int_{s_0}^t \|H_u(b)\|_2^2 du\}^{1/2} \sqrt{t - s_0}.$$

Recall that $H_u(b) = dE_u(E_u(b)) - E_u(dE_u(b))$. Using the formula in Proposition 2.6, $dE_u(b) = dE_u(E_u(b)) + E_u(dE_u(b))$, one obtains that

$$H_u(b) = dE_u(b) - 2E_u(dE_u(b)) = (1 - 2E_u)(dE_u(b)).$$

Note that E_u is (or rather, extends to) a self adjoint projection in $\mathcal{H}$. Therefore $1 - 2E_u$ is a symmetry, i.e. a selfadjoint unitary operator. In particular, it is $\| \|_2$-isometric. Therefore

$$\|H_u(b)\|_2 = \|(1 - 2E_u)(dE_u(b))\|_2 = \|dE_u(b)\|_2.$$

Then (using Hypothesis (1))

$$| < S_{n+1}(t) - S_n(t), b > | \leq \sup_{u \in [s_0,t]} \|S_n(u) - S_{n-1}(u)\|_2 \{ \int_{s_0}^t \|dE_u(b)\|_2^2 du\}^{1/2} \sqrt{t - s_0}$$

$$\leq \sup_{u \in [s_0,t]} \|S_n(u) - S_{n-1}(u)\|_2 C^{1/2} \|b\|_2 \sqrt{t - s_0}.$$

Taking supremum over $b \in \mathcal{A}$ with $\|b\|_2 = 1$ proves the inequality. □

Corollary 3.5. *Fix $s_0 \in I$ and $a \in \mathcal{A}$. If Hypothesis (1) holds, then there exists $t_0 \in I$, $s_0 < t_0$, such that the sequence $S_n^{s_0,a}(t) = S_n(t)$ converges uniformly in the norm $\| \ \|_2$, in the interval $[s_0, t_0]$, to a function $S(t)$. This function $S(t)$ takes values in $\mathcal{A}$, is uniformly $\| \ \|_\infty$-bounded, and weakly continuously differentiable as an $\mathcal{H}$-valued map. Moreover, for $t \in [s_0, t_0]$ and $\xi \in \mathcal{H}$,*

$$< S(t), \xi > = < a, \xi > + \int_{s_0}^t < H_s(S(s)), \xi > ds.$$

Proof. Pick t_0 such that $k_0 = C^{1/2}\sqrt{t_0 - s_0} < 1$, where C is the constant in the above Proposition. Then, if $t \in [s_0, t_0]$,

$$\|S_{n+1}(t) - S_n(t)\|_2 \leq C^{1/2}\sqrt{t - s_0} \sup_{u \in [s_0,t]} \|S_n(u) - S_{n-1}(u)\|_2$$

$$\leq C^{1/2}\sqrt{t_0 - s_0} \sup_{u \in [s_0,t_0]} \|S_n(u) - S_{n-1}(u)\|_2 = k_0 \sup_{u \in [s_0,t_0]} \|S_n(u) - S_{n-1}(u)\|_2.$$

Then

$$\sup_{t \in [s_0,t_0]} \|S_{n+1}(u) - S_n(t)\|_2 \leq k_0 \sup_{t \in [s_0,t_0]} \|S_n(t) - S_{n-1}(t)\|_2.$$

It follows, by a well-known argument, that $S_n(t)$ converges in $\mathcal{H}$ to a function $S(t)$, uniformly in $[s_0, t_0]$. The maps $S_n(t)$ are $\mathcal{A}$-valued and uniformly $\| \ \|_\infty$-bounded in $[s_0, t_0]$, therefore $S(t)$ is also $\mathcal{A}$-valued, and uniformly $\| \ \|_\infty$-bounded. Note that it is weakly continuous as an $\mathcal{H}$-valued map: if $\xi \in \mathcal{H}$, then $< S(t+h) - S(t), \xi >$ equals

$$< S(t+h) - S_n(t+h), \xi > + < S_n(t+h) - S_n(t), \xi > + < S_n(t) - S(t), \xi > .$$

and the proof follows by a typical $\epsilon/3$ argument. Finally, by construction, for any $x \in \mathcal{A}$

$$< S_{n+1}(t), x > = < a, x > + \int_{s_0}^t < H_u(S_n(u)), x > du = < a, x > - \int_{s_0}^t < S_n(u), H_u(x) > du.$$

Note that $< S_n(u), H_u(x) >$ tends uniformly to $< S(u), H_u(x) >$ in the interval $[s_0, t_0]$. Indeed,

$$| < S_n(u), H_u(x) > - < S(u), H_u(x) > | \leq \|S_n(u) - S(u)\|_2 \|H_u(x)\|_2$$

$$\leq \|S_n(u) - S(u)\|_2 \|H_u(x)\|_\infty,$$

where, as seen before, $\|H_u(x)\|_\infty$ is uniformly bounded in $[s_0, t_0]$. Therefore, in the expression above, taking limit $n \to \infty$, one obtains

$$< S(t), x > = < a, x > + \int_{s_0}^t < H_u(S(u)), x >$$

for all $x \in \mathcal{A}$. By density, it follows that

$$< S(t), \xi >=< a, \xi > + \int_{s_0}^{t} < H_u(S(u)), \xi >$$

for all $\xi \in \mathcal{H}$. In particular, this implies that $S(t)$ is weakly continuously differentiable as an $\mathcal{H}$-valued map. $\square$

The next step is to extend this weak solution. Fix a closed bounded interval $J_0 \subset I$, and let $C = C_{J_0}$ be the constant in the inequality of Hypothesis (1) for this sub-interval. If $s_0 \in J_0$, then the length of the interval $[s_0, t_0]$ on which a solution is defined depends only on this constant C. It does not depend on the initial condition a. It follows that one can glue solutions in a standard fashion, to obtain a solution $S(t)$ defined in the whole sub-interval J_0. Uniqueness of solutions follows. Indeed, suppose that S_1, S_2 are two solutions with $S_1(s) = S_2(s)$. Then

$$S_i(t) = a + \textbf{weak} \int_{s}^{t} H_u(S_i(u))du \quad i = 1, 2.$$

Thus, as in Proposition 3.4,

$$\|S_1(t) - S_2(t)\|_2 \leq C_{J_0}^{1/2} \sqrt{t-s} \sup_{u \in [s,t]} \|S_1(t) - S_2(t)\|_2.$$

Then S_1 and S_2 coincide up to time t such that $|t - s| < 1/C_{J_0}$. Note that this constant does not depend on s. It follows that S_1 and S_2 coincide in J_0. Clearly this holds on any closed bounded sub-interval $J_0 \subset I$.
Let us summarize these results.

Theorem 3.6. *Suppose that Hypothesis (1) holds. Let $a \in \mathcal{A}$, Then there exists a map $\alpha_s(t)$, which is $\mathcal{A}$-valued, uniformly $\| \|_\infty$-bounded on closed bounded subintervals of I, and weakly continuously differentiable as an $\mathcal{H}$-valued function, which is the unique (weak) solution of the transport equation (2)*

$$\begin{cases} \dot{\alpha}(t) = [dE_t, E_t](\alpha(t)) \\ \alpha(s) = a. \end{cases}$$

Remark 3.7. For $s, t \in I$, denote by $G_{t,s}$ the propagator of the transport equation, i.e.

$$G_{t,s} : \mathcal{A} \to \mathcal{A} \ , \quad G_{t,s}(a) = \alpha_s(t),$$

where α_s is the solution of (2) with $\alpha_s(s) = a$. The propagator has the following properties:

1. $G_{t,s}$ is isometric for the $\| \|_2$ norm: $\|G_{t,s}(a)\|_2 = \|a\|_2$.
2. For each $a \in \mathcal{A}$, $G_{t,s}(a)$, as an $\mathcal{H}$-valued map, is weakly continuously differentiable in the parameter t, and continuous in the parameter s.
3. $G_{s,s}(a) = a$, for all $a \in \mathcal{A}$.
4. $G_{t,s}G_{s,r} = G_{t,r}$.

To prove the first assertion, put $\alpha_s(t) = G_{t,s}(a)$, ($\alpha_s(s) = a$), then

$$\frac{d}{dt} < G_{t,s}(a), G_{t,s}(a) >=< H_t(\alpha_s(t)), \alpha_s(t) > + < \alpha_s(t), H_t(\alpha_s(t)) >= 0.$$

Here we use the fact that the product rule holds for weak solutions because they are uniformly $\| \|_\infty$-bounded, and also that $H_t = [dE_t, E_t]$ is anti-symmetric. Therefore

$$\|G_{t,s}(a)\|_2^2 = \|G_{s,s}(a)\|_2^2 = \|a\|_2^2.$$

The third and fourth assertions are apparent. To prove the second, use the fourth:

$$G_{t,s+h}(a) - G_{t,s}(a) = G_{t,s}(G_{s,s+h}(a) - a).$$

And then, for $b \in \mathcal{A}$,

$$< G_{t,s+h}(a) - G_{t,s}(a), b >=< G_{s,s+h}(a) - a, G^*_{t,s}(b) >$$

$$= \int_s^{s+h} < H_u(G_{u,s+h}(a) - a), G^*_{t,s}(b) > du.$$

For $|h| < \delta$ such that $[s - \delta, s + \delta] \subset I$ there exists a constant D such that $\|dE_u\|_{\infty,\infty} \leq D$. Then

$$\|H_u(G_{u,s+h}(a) - a)\|_2 = \|dE_u(G_{u,s+h}(a) - a)\|_2 \leq \|dE_u(G_{u,s+h}(a) - a)\|_\infty$$

$$\leq D\|G_{u,s+h}(a) - a\|_\infty,$$

which is uniformly bounded for such h, by a constant D'. Therefore

$$| < G_{t,s+h}(a) - G_{t,s}(a), b > | \leq | \int_s^{s+h} | < H_u(G_{u,s+h}(a) - a), G^*_{t,s}(b) > | \quad du|$$

$$\leq | \int_s^{s+h} \|H_u(G_{u,s+h}(a) - a)\|_2 \|b\|_2 du| \leq D'|h|\|b\|_2.$$

Taking supremum over $b \in \mathcal{A}$ with $\|b\|_2 = 1$, one has

$$\|G_{t,s+h}(a) - G_{t,s}(a)\|_2 \leq D'|h|.$$

Note that one obtains more than continuity in the parameter s.
In particular, these facts imply that the map

$$G_t : \mathcal{A} \to \mathcal{A} \ , \quad G_t := G_{t,0} \tag{3}$$

is invertible, its inverse is $G_t^{-1} = G_{0,t}$.

4 The propagators as intertwiners

In this section we show that the linear isomorphisms G_t intertwine the expectations:

$$G_t \circ E_0 \circ G_t^{-1} = E_t.$$

To this effect, the following result is needed.

Proposition 4.1. *Let $\alpha(t)$, $t \in I$ be a (weak) solution of the transport equation (2). Then the map $E_t(\alpha(t))$ is also a solution. In particular, if at any given instant $t_0 \in I$ one has that $\alpha(t_0) \in \mathcal{B}_{t_0}$, then $\alpha(t) \in \mathcal{B}_t$ for all $t \in I$.*

Proof. First we must show that $\beta = E(\alpha)$ is $\mathcal{A}$-valued, $\| \ \|_\infty$-bounded and weakly continuously differentiable as an $\mathcal{H}$-valued function. The first fact is apparent. The second: $\|E_t(\alpha(t))\|_\infty \leq \|\alpha(t)\|_\infty$. The third: if $\xi \in \mathcal{H}$

$$\frac{1}{h} < \beta(t+h) - \beta(t), \xi >=< E_{t+h}(\frac{\alpha(t+h) - \alpha(t)}{h}), \xi > + < (\frac{E_{t+h} - E_t}{h})(\alpha(t)), \xi > .$$

The second term tends to $< dE_t(\alpha(t)), \xi >$ as $h \to 0$, by definition. For the first term we can aply Lemma 2.5, and it follows that it tends to $< E_t(\dot{\alpha}(t)), \xi >$. Then $E(\alpha)$ is weakly differentiable, and its derivative is $dE(\alpha) + E(\dot{\alpha})$, which is weakly continuous. Let us verify that $E(\alpha)$ is a solution:

$$\frac{d}{dt}E(\alpha) = dE(\alpha) + E(\dot{\alpha}) = dE(\alpha) + E(dE(E(\alpha))) - E(E(dE(\alpha))).$$

Recall from Lemma 2.6 that $dE = dE(E) + E(dE)$, which in particular implies that

$$E(dE)E = 0.$$

Then the expression above equals

$$dE(\alpha) - E(dE(\alpha)) = dE(E(\alpha)).$$

On the other hand

$$[dE, E](E(\alpha)) = dE(E(E\alpha)) - E(dE(E(\alpha))) = dE(E(\alpha)).$$

The last assertion follows by uniqueness of solutions. □

Our main result follows:

Theorem 4.2. *Let $E_t : \mathcal{A} \to \mathcal{B}_t \subset \mathcal{A}$, $t \in I$ be a curve of trace invariant conditional expectations, such that for each $x \in \mathcal{A}$ and $\xi \in \mathcal{H}$, the $\mathcal{H}$-valued curve $E_t(x)\xi$ is continuously differentiable. Suppose also that E_t verifies Hypothesis (1), i.e. for each closed bounded subinterval $J \subset I$,*

$$\int_J \|dE_t(a)\|_2^2 dt \le C_J \|a\|_2^2.$$

Then the curve of propagators $G_t : \mathcal{A} \to \mathcal{A}$, $t \in I$, verifies:

1. *For each $a \in \mathcal{A}$, the curve $I \ni t \to G_t(a) \in \mathcal{A} \subset \mathcal{H}$ is weakly continuously differentiable, with $G_0 = Id$.*
2. *The maps G_t are unital and $*$-preserving.*
3. *For each $t \in I$,*
$$G_t E_0 G_t^{-1} = E_t.$$

Proof. The first assertion is apparent: $G_t(a)$ is a weak solution of the transport equation. Since $E_t(1) = 1$ for all t, $dE_t(1) = 0$, and therefore $H_t(1) = 0$. Therefore $\alpha(t) = 1$ for all t is a solution, i.e. $G_t(1) = 1$. The maps E_t are also $*$-preserving: $E_t(a^*) = E_t(a)^*$, therefore also $dE_t(a^*) = dE_t(a)^*$ and $H_t(a^*) = H_t(a)^*$. Therefore if $\alpha(t)$ is a solution, then also $\alpha^*(t)$ is a solution, and thus $G_t(a^*) = G_t(a)^*$. For the last assertion, note that by the above Proposition, $E_t(G_t(a))$ is a solution. Clearly also $G_t(E_0(a))$ is a solution. At $t = 0$, they take the values $E_0(G_0(a)) = E_0(a)$ and $G_0(E_0(a)) = E_0(a)$, therefore $E_t(G_t(a)) = G_t(E_0(a))$ for all $t \in I$. □

Remark 4.3. Under the hypothesis of the above theorem, the first assertion in Remark 3.7 implies that the propagators $G_t : \mathcal{A} \to \mathcal{A}$ can be extended to unitary operators U_t acting in $\mathcal{H}$. Clearly they preserve $\mathcal{A} \subset \mathcal{H}$: $U_t(\mathcal{A}) \subset \mathcal{A}$. Moreover, if e_t denotes the extension of E_t to an operator in $\mathcal{H}$, in fact a selfadjoint projection, the last assertion implies that these projections are unitarily equivalent, more precisely

$$U_t e_0 U_t^* = e_t \ , \quad t \in I.$$

The identity $G_t E_0 G_t^{-1} = E_t$ of the above theorem, in particular implies that G_t maps $\mathcal{B}_0$ onto $\mathcal{B}_t$. Our next result shows that this restriction is a multiplicative $*$-isomorphism.

Theorem 4.4. *Assume Hypothesis (1). Then for each $t \in I$, the map $\theta_t := G_t|_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathcal{B}_t$ is a multiplicative $*$-isomorphism.*

Proof. The above indentity clearly implies that $\theta_t(\mathcal{B}_0) = \mathcal{B}_t$. Also it is clear that θ_t is linear, $*$-preserving and bijective. Thus it only remains to prove that it is multiplicative. Let $a, b \in \mathcal{B}_0$, and denote by α and β the solutions of the transport equation with $\alpha(0) = a$ and $\beta(0) = b$. Note that Proposition 4.1 implies that both $\alpha(t), \beta(t) \in \mathcal{B}_t$, i.e. $E_t(\alpha(t)) = \alpha(t)$, $E_t(\beta(t)) = \beta(t)$. Let $x \in \mathcal{A}$. Differentiating the identity

$$< E_t(\alpha(t)), x >=< \alpha(t), x >$$

one obtains

$$< dE_t(\alpha(t)), x > + < E_t(\dot\alpha(t)), x >=< \dot\alpha(t), x > .$$

This last term equals $< [dE_t, E_t](\alpha(t)), x >$. Note that

$$E_t(dE_t(\alpha(t))) = E_t(dE_t(E_t(\alpha(t)))) = 0.$$

Therefore

$$< [dE_t, E_t](\alpha(t)), x >=< dE_t(\alpha(t)), x > .$$

Then $< E_t(\dot\alpha(t)), x >= 0$, i.e. $E_t(\dot\alpha(t)) = 0$. Conversely, if a map $\gamma(t)$ takes values in $\mathcal{B}_t$ and verifies $E_t(\dot\gamma(t)) = 0$, then it is a solution of the transport equation.
The curve $\alpha(t)\beta(t)$ takes values in $\mathcal{B}_t$. Also it is clear that the product rule applies for the derivative of $\alpha(t)\beta(t)$ (as they are $\| \|_\infty$ uniformly bounded on closed bounded intervals). Then

$$E_t(\frac{d}{dt}(\alpha(t)\beta(t))) = E_t(\dot\alpha(t)\beta(t)) + E_t(\alpha(t)\dot\beta(t)) = E_t(\dot\alpha(t))\beta(t) + \alpha(t)E_t(\dot\beta(t)) = 0,$$

i.e. $\alpha(t)\beta(t)$ is a solution of the transport equation, with initial condition ab. It follows that

$$\theta_t(ab) = G_t(ab) = \alpha(t)\beta(t) = \theta_t(a)\theta_t(b).$$

□

It was shown above that a solution that starts in $R(E_0) = \mathcal{B}_0$, remains in $R(E_t) = \mathcal{B}_t$ at time t. The intertwining identity implies that the same is true for the kernels: if $E_0(a) = 0$, then $E_t(\alpha(t)) = 0$. In other words, if $a \in \mathcal{A}$ is decomposed as

$$a = b + z \quad b \in \mathcal{B}_0 \text{ and } E_0(z) = 0,$$

putting $\beta(t) = G_t(b)$ and $z(t) = G_t(z)$ the solutions with initial conditions b and z, then

$$\alpha(t) = \beta(t) + z(t) \quad \beta(t) \in \mathcal{B}_t \text{ and } E_t(z(t)) = 0,$$

which is an orthogonal decomposition. The next result shows that their derivatives are also orthogonal for all t, though the role of the subspaces is reversed.

Proposition 4.5. *With the above notations, $E_t(\dot\beta(t)) = 0$ and $\dot z(t) \in \mathcal{B}_t$*

Proof. As it was shown in the proof of the previous theorem, the solution $\beta(t)$ verifies $\dot\beta(t) = dE_t(\beta(t))$, as well as $E_t(dE_t(\beta(t))) = 0$. Putting these two together gives $E_t(\dot\beta(t)) = 0$.
On the other hand, since $E_t(z(t)) = 0$,

$$\dot z(t) = [dE_t, E_t](z(t)) = E_t(dE_t(z(t))),$$

i.e. $\dot z(t) \in \mathcal{B}_t$. □

5 Systems of projections

Let $\mathsf{p} = (p_1, p_2, \ldots)$ be a (finite or infinite) system of projections in $\mathcal{A}$, i.e. a sequence of pairwise orthogonal projections which strongly sum 1. Such a system gives rise to a conditional expectation:

$$E_{\mathsf{p}} : \mathcal{A} \to \mathcal{B} \subset \mathcal{A}, \quad E_{\mathsf{p}}(x) = \sum_{i \geq 1} p_i x p_i.$$

The range of this conditional expectation is the sub-algebra $\mathcal{B}$ of elements of $\mathcal{A}$ which commute with all p_i, $i \geq 1$. Suppose that a curve $\mathsf{p}(t) = (p_1(t), p_2(t), \ldots)$, $t \in I$ of systems of projections is given, and that it satisfies that

$$I \ni t \mapsto p_i(t)\xi \in \mathcal{H}$$

is C^1 for all $\xi \in \mathcal{H}$ and every $i \geq 1$. We shall examine the meaning of the smoothness condition on the curve $E_t = E_{\mathsf{p}(t)}$. We show that if $t \mapsto E_t(a)\xi$ is continuously differentiable (for any $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$), then Hypothesis (1) holds.
Our first elementary observation is that if the system is finite, then these conditions are fulfilled.

Proposition 5.1. *Suppose that the system $\mathsf{p}(t)$ is finite, i.e. $\mathsf{p}(t) = (p_1(t), \ldots, p_n(t))$, and that for each $j = 1, \ldots, n$, the curve $p_j(t)\xi$ is C^1 in $\mathcal{H}$. Then curve E_t verifies that $E_t(a)\xi$ is C^1 in $\mathcal{H}$ for each $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, and dE_t is bounded in $\mathcal{H}$.*

Proof. Pick $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$. Then $E_t(a)\xi$ is C^1. Indeed, a straightforward computation shows that the product rule holds and that

$$\frac{d}{dt} E_t(a)\xi = \sum_{i=1}^{n} \dot{p}_i(t) a p_i(t)\xi + p_i(t) a \dot{p}_i(t)\xi.$$

This map is clearly continuous. Next note that for each j, the map $\xi \mapsto \dot{p}_j(t)\xi$ is linear and everywhere defined in $\mathcal{H}$. Moreover, it is symmetric:

$$< \dot{p}_j\xi, \eta >= \frac{d}{dt} < p_j(t)\xi, \eta >= \frac{d}{dt} < \xi, p_j(t)\eta >=< \xi, \dot{p}_j(t)\eta > .$$

Therefore, by the closed graph theorem, it is a bounded operator. Since it is defined as a strong limit, it takes values in $\mathcal{A}$, i.e. $\dot{p}_j \in \mathcal{A}$. The operator dE_t coincides in $\mathcal{A}$ with

$$\sum_{i=1}^{n} L_{\dot{p}_i(t)} R_{p_i(t)} + L_{p_i(t)} R_{\dot{p}_i(t)},$$

which is clearly bounded (Here L_a, R_a denote left and right multiplication by $a \in \mathcal{A}$). Moreover, by the uniform boundedness principle, for $t \in J \subset I$, a closed a bounded sub-interval, the norms $\|\dot{p}_j(t)\|_\infty$ are uniformly bounded by C (which can be chosen independent of j as well). Therefore it is apparent that dE_t is bounded in $\mathcal{H}$:

$$\|dE_t(a)\|_2 \leq nC\|a\|_2, \ t \in J.$$

□

We restrict now to infinite systems. First we discuss a condition which implies the regularity of the curve E_t. Namely the following, which was studied in [1] for expectations in the algebra of compact operators.

Definition 5.2. *We shall say that the curve of systems of projections* $\mathsf{p}(t)$ *has square summable derivatives if for every closed bounded subinterval* $J \subset I$*, there exists a constant* D_J *such that*

$$\sum_{i\geq 1} \|\dot{p}_i(t)\xi\|_2^2 \leq D_J \|\xi\|_2^2 \tag{4}$$

for every $\xi \in \mathcal{H}$ *and* $t \in J$*.*

Proposition 5.3. *The curve* $\mathsf{p}(t)$ *has square summable derivatives (4) if and only if there exists a strongly* C^1 *curve* u_t*,* $t \in I$*, of unitary operators in* $\mathcal{A}$ *such that* $p_i(t) = u_t p_i(0) u_t^*$ *for all* $i \geq 1$*.*

Proof. Suppose first that inequality (4) holds. Then we claim that for any $\xi \in \mathcal{H}$ the series

$$\sum_{i\geq 1} p_i(t)\dot{p}_i(t)\xi$$

is convergent in $\mathcal{H}$. Indeed, note that since the vectors $p_i(t)\dot{p}_i(t)\xi$ are pairwise orthogonal,

$$\| \sum_{i\geq N+1} p_i(t)\dot{p}_i(t)\xi\|_2^2 = \sum_{i\geq N+1} \|p_i(t)\dot{p}_i(t)\xi\|_2^2 \leq \sum_{i\geq N+1} \|\dot{p}_i(t)\xi\|_2^2,$$

which tends to 0 as N goes to ∞. Then this series produces an everywhere defined linear operator

$$\Delta_t \xi = \sum_{i\geq 1} p_i(t)\dot{p}_i(t)\xi.$$

This operator has an everywhere defined adjoint, given by the series

$$\Delta^* \xi = \sum_{i\geq 1} \dot{p}_i(t)p_i(t)\xi,$$

which is weakly convergent in $\mathcal{H}$:

$$< \Delta^*\xi, \eta >= \sum_{i\geq 1} < \dot{p}_i(t)p_i(t)\xi, \eta >= \sum_{i\geq 1} < \xi, p_i(t)\dot{p}_i(t)\eta > .$$

Therefore, by the closed graph theorem, Δ_t is bounded, and since it is defined as a strong limit of elements of $\mathcal{A}$, $\Delta_t \in \mathcal{A}$. Note that the identity $\dot{p}_i(t) = \dot{p}_i(t)p_i(t) + p_i(t)\dot{p}_i(t)$ implies that, since $\sum_{i\geq 1} p_i(t)\xi = \xi$ and this series converges uniformly in closed bounded sub-intervals,

$$0 = \frac{d}{dt} \sum_{i\geq 1} < p_i(t)\xi, \eta >= \sum_{i\geq 1} < \dot{p}_i(t)\xi, \eta >= \sum_{i\geq 1} < \dot{p}_i(t)p_i(t) + p_i(t)\dot{p}_i(t)\xi, \eta >$$

$$=< \Delta_t^*\xi + \Delta_t\xi, \eta >,$$

i.e. Δ_t is anti-hermitic. Furthermore, the hypothesis that the curve $\mathsf{p}(t)$ has square summable derivatives (4), implies that on closed bounded sub-intervals, the series that defines Δ_t is uniformly convergent. Therefore the map

$$I \ni t \mapsto \Delta_t \xi \in \mathcal{H}$$

is continuous, that is $t \mapsto \Delta_t \in \mathcal{A}$ is strongly continuous. For any $\xi_0 \in \mathcal{H}$, consider the linear differential equation in $\mathcal{H}$

$$\begin{cases} \dot{\mu}(t) = -\Delta_t \mu(t) \\ \mu(0) = \xi_0. \end{cases} \tag{5}$$

It was shown in [1] in a different context, that the unitary propagator u_t of this equation, (defined by $u_t\xi_0 = \mu(t)$), verifies

$$u_t p_i(0) u_t^* = p_i(t), \quad i \geq 1.$$

The computation is formally identical in this context, and thus these relations hold. Moreover, apparently $u_t \in \mathcal{A}$, and the map $t \mapsto u_t\xi_0$ is C^1 for every $\xi_0 \in \mathcal{H}$.
Conversely, suppose the existence of a strongly C^1 curve u_t of unitaries in $\mathcal{A}$ such that $u_t p_i(0) u_t^* = p_i(t)$ for $i \geq 1$. Then the product rule holds and

$$\dot{p}_i(t)\xi = \dot{u}_t p_i(0) u_t^* \xi + u_t p_i(0) \dot{u}_t^* \xi.$$

Then $\|\dot{p}_i(t)\xi\|_2 \leq \|\dot{u}_t p_i(0) u_t^*\xi\|_2 + \|p_i(0)\dot{u}_t^*\xi\|_2$. Note that for any closed bounded subinterval $J \subset I$, the family of vectors $\{\dot{u}_t\xi : t \in J\}$ is uniformly bounded. Therefore, by the uniform boundedness principle, $\|\dot{u}_t\| \leq K_J$ for all $t \in J$. Then, using that $p_i(0)$ are pairwise orthogonal and sum 1,

$$\sum_{i\geq 1} \|\dot{u}_t p_i(0) u_t^*\xi\|_2^2 \leq K_J^2 \sum_{i\geq 1} \|p_i(0) u_t^*\xi\|_2^2 = K_J^2 \|u_t^*\xi\|_2^2 = K_J^2 \|\xi\|_2^2,$$

and

$$\sum_{i\geq 1} \|p_i(0)\dot{u}_t^*\xi\|_2^2 = \|\dot{u}_t^*\xi\|_2^2 \leq K_J^2 \|\xi\|_2^2.$$

Then

$$\sum_{i\geq 1} \|\dot{p}_i(t)\xi\|_2^2 \leq 4K_J^2 \|\xi\|_2^2,$$

for $t \in J$. □

Remark 5.4. Note that (under the assumption (4) that the system of projections has square summable derivatives), the unitaries u_t provide anoter way to intertwine E_0 and E_t. Indeed, put $\Omega_t = Ad(u_t)$ ($\Omega_t(x) = u_t x u_t^*$), then

$$\Omega_t E_0 \Omega_t^{-1}(x) = u_t \sum_{i\geq 1} u_t p_i(0) u_t^* x u_t p_i(0) u_t^* = \sum_{i\geq 1} p_i(t) x p_i(t) = E_t(x).$$

We shall consider the relation between Ω_t and G_t below. Our purpose now is to use this inner automorphisms to prove the regularity of the curve E_t. To this effect, note that for each $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, the map $I \ni t \mapsto \Omega_t(a)\xi$ is C^1. Indeed,

$$\frac{1}{h}\{u_{t+h} a u_{t+h}^* \xi - u_t a u_t^* \xi\} = \frac{1}{h}\{u_{t+h} a (u_{t+h}^*\xi - u_t^*\xi)\} + \frac{1}{h}\{u_{t+h} a u_t^*\xi - u_t a u_t^*\xi\}.$$

The second term tends to $\dot{u}_t a u_t^*\xi$ as $h \to 0$, because u_t is strongly C^1. The first term tends to $u_t a \dot{u}_t^*\xi$. Indeed, $\|\frac{1}{h}\{u_{t+h} a (u_{t+h}^*\xi - u_t^*\xi)\} - u_t a \dot{u}_t^*\xi\|_2$ is bounded by

$$\|u_{t+h} a \frac{1}{h}\{u_{t+h}^*\xi - u_t^*\xi\} - u_{t+h} a \dot{u}_t^*\xi\|_2 + \|u_{t+h} a \dot{u}_t^*\xi - u_t a \dot{u}_t^*\xi\|_2$$

$$\leq \|a \frac{1}{h}\{u_{t+h}^*\xi - u_t^*\xi\} - a\dot{u}_t^*\xi\|_2 + \|u_{t+h}\eta - u_t\eta\|_2,$$

where $\eta = a\dot{u}_t^*\xi$. Clearly both terms tend to 0. Finally, the derivative of $\Omega_t(a)\xi$ equals

$$\dot{\Omega}_t(a)\xi = \dot{u}_t a u_t^*\xi + u_t a \dot{u}_t^*\xi,$$

which is clearly continuous.

Next we show that condition (4) guarantees that equation (2) has existence and uniqueness of solutions.

Proposition 5.5. *If the system of projections* $\mathsf{p}(t)$ *has square summable derivatives (4), then the map* $I \ni t \mapsto E_t(a)\xi \in \mathcal{H}$ *is* C^1*. Moreover, the derivative* dE_t *extends to a bounded operator in* $\mathcal{H}$*.*

Proof. As seen above, $E_t(x) = \Omega_t(E_0(\Omega_t^{-1}(x)))$. Note that for each $x \in \mathcal{A}$, both $\Omega_t(x)$ and $\Omega_t^{-1}(x) = u_t^* x u_t$ are strongly C^1. Then for each $x \in \mathcal{A}$ and $\xi \in \mathcal{H}$,

$$\frac{1}{h}\{E_{t+h}(x)\xi - E_t(x)\xi\} = \Omega_{t+h}E_0(\frac{1}{h}\{\Omega_{t+h}(x) - \Omega_t(x)\})\xi + \frac{1}{h}\{(\Omega_{t+h}(x)\eta - \Omega_t(x)\eta\},$$

where $\eta = E_0(\Omega_t^{-1}(x))\xi$. The first term tends to $\Omega_t E_0 \dot{\Omega}_t(x)\xi$: put

$$b_h = E_0(\frac{1}{h}\{\Omega_{t+h}(x) - \Omega_t(x)\}),$$

which tends strongly to $b_0 = E_0(\dot{\Omega}_t(x))$ (because E_0 is strongly continuous), then

$$\|\Omega_{t+h}(b_h)\xi - \Omega_t(b_0)\xi\|_2 \le \|\Omega_{t+h}(b_h)\xi - \Omega_t(b_h)\xi\|_2 + \|\Omega_t(b_h)\xi - \Omega_t(b_0)\xi\|_2.$$

The second term clearly tends to 0. The first term is bounded by

$$\|u_{t+h}b_h(u_{t+h}^* - u_t^*)\xi\|_2 + \|u_t b_h(u_{t+h}^* - u_t^*)\xi\|_2 \le 2\|b_h\|_\infty \|u_{t+h}^*\xi - u_t^*\xi\|_2.$$

This term tends to zero because the involution $*$ is strongly continuous ($\mathcal{A}$ is finite) and $\|b_h\|_\infty$ is bounded for $|h|$ small:

$$\|b_h\|_\infty \le \|\frac{1}{h}\{\Omega_{t+h}(x) - \Omega_t(x)\}\|_\infty,$$

with $\frac{1}{h}\{\Omega_{t+h}(x) - \Omega_t(x)\}$ strongly convergent, and therefore locally $\| \ \|_\infty$-bounded.
Note that, in the above notations, $\xi \mapsto \dot{u}_t\xi$ is an everywhere defined operator. Clearly $u_t^*\dot{u}_t$ is anti-hermitian:

$$0 = \frac{d}{dt} < u_t\xi, u_t\eta > = < u_t^*\dot{u}_t\xi, \eta > + < \xi, u_t^*\dot{u}_t\eta > .$$

Then, by the closed graph theorem, $u_t^*\dot{u}_t$ is bounded, and therefore $\dot{u}_t$ is bounded. Also it is clear that, being a strong limit of operators in $\mathcal{A}$, it belongs to $\mathcal{A}$. Then

$$\dot{\Omega}_t = L_{\dot{u}_t}R_{u_t^*} + R_{\dot{u}_t}L_{u_t^*}$$

is bounded. Also it is clear that $\Omega_t^{-1} = Ad(u_t^*)$ has the same properties. Then

$$dE_t = \dot{\Omega}_t E_0 \Omega_t^{-1} + \Omega_t E_0 \dot{\Omega^{-1}}_t$$

is bounded in $\mathcal{H}$. □

Remark 5.6. In [1], similar results were obtained for the algebra $\mathcal{K}(\mathcal{H})$ of compact operators. For instance it was shown that if the systems $\mathsf{p}(t)$ consist of more than two projectors, then Ω_t and G_t differ. It was also shown that they coincide if the system consists of two projections, and that Ω_t and G_t coincide in $\mathcal{B}_0$. In other words, always under the assumption that inequality (4) holds, the unitaries u_t of $\mathcal{A}$ which solve equation (5), implement the automorphism θ_t:

$$\theta_t = Ad(u_t)|_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathcal{B}_t.$$

We refer the reader to [1] for the proofs of these facts, which though perfomed in $\mathcal{K}(\mathcal{H})$, are formally identical in our situation.

We now show that for this class of conditional expectations, given by a system of projections, smoothness of the curve E_t implies Hypothesis (1).

Proposition 5.7. *Let* $\mathsf{p}\,(t)$*,* $t \in I$*, be a system of projectors and* E_t *as above, verifying that* $I \ni t \mapsto E_t(a)\xi \in \mathcal{H}$ *is* C^1*, for every* $a \in \mathcal{A}$ *and* $\xi \in \mathcal{H}$*, and that for each* $j \geq 1$*,* $t \mapsto p_j(t)\xi$ *is* C^1*. Then Hypothesis (1) holds: for each closed and bounded sub-interval* $J \subset I$*, there exists* C_J *such that*

$$\int_J \|dE_t(a)\|_2^2 dt \leq C_J \|a\|_2^2$$

for each $a \in \mathcal{A}$*.*

Proof. Note that the map $t \mapsto p_j(t) \in \mathcal{A}$ is a solution of equation (2). Since $p_j(t) \in \mathcal{B}_t$, this equation becomes simpler, as seen in the previous section. Namely, one has to show that

$$\dot{p}_j(t) = dE_t(p_j(t)).$$

Indeed:

$$dE_t(p_j(t)) = \sum_{i \geq 1} \dot{p}_i(t)p_j(t)p_i(t) + p_i(t)p_j(t)\dot{p}_i(t) = \dot{p}_j(t)p_j(t) + p_j(t)\dot{p}_j(t) = \dot{p}_j(t),$$

where the last identity follows from differentiating $p_j(t)p_j(t) = p_j(t)$. Then we can bound the operator norm of $\dot{p}_j(t) \in \mathcal{A}$:

$$\|\dot{p}_j(t)\|_\infty = \|dE_t(p_j(t))\|_\infty \leq \|dE_t\|_{\infty,\infty} \leq D_J$$

for a constant D_J independent of $t \in J$. Then

$$\|\sum_{i \geq 1} \dot{p}_i(t)ap_i(t)\|_2^2 = \sum_{i \geq 1} \tau(p_i(t)a^*(\dot{p}_i(t))^2 ap_i(t)) \leq D_J^2 \sum_{i \geq 1} \tau(p_i(t)a^*ap_i(t))$$

$$= D_J^2 \sum_{i \geq 1} \tau(p_i(t)a^*a) = D_J^2 \tau(a^*a) = D_J^2 \|a\|_2^2.$$

Analogously, $\|\sum_{i \geq 1} p_i(t)a\dot{p}_i(t)\|_2^2 \leq D_J^2 \|a\|_2^2$. Then

$$\|dE_t(a)\|_2^2 = \|\sum_{i \geq 1} \dot{p}_i(t)ap_i(t) + \sum_{i \geq 1} p_i(t)a\dot{p}_i(t)\|_2^2 \leq 4D_j^2 \|a\|_2^2.$$

Therefore

$$\int_J \|dE_t(a)\|_2^2 dt \leq 4|J|D_J^2 \|a\|_2^2.$$

$\square$

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